INFINITE LOOP SPACES ASSOCIATED TO AFFINE KAC-MOODY GROUPS

XIANZU LIN

Institute of Mathematics, Academy of Mathematics and Systems Science Chinese Academy of Sciences, Beijing, 100190, China
Email: linxianzu@126.com

Abstract. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups, It is well known that to each infinite class of classical groups over a commutative ring R, we can associate an infinite loop space $G(R)$ by Quillen's plus construction, in fact it is a functor from the category of commutative rings to the category of infinite loop spaces. In this paper we generalize this fact to the cases of affine Kac-Moody groups. Roughly speaking, there are seven infinite classes of affine Kac-Moody groups, and to each infinite class we can associate an analogous functor.

Keywords: generalized Cartan matrix; infinite loop space; affine Kac-Moody group;

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1. Introduction

We say that a pointed space X is an *infinite loop space* if there is a sequence of (pointed) spaces $X_0, X_1, \cdots$ with $X_0 = X$ and weak homotopy equivalences $X_n \simeq \Omega X_{n+1}$.

Example 1.1. *Let $GL(n)$ be the general linear group over $\mathbb{C}$ and let BGL be the limit of classifying space $\varinjlim BGL(n)$. By the Bott peoriodicity theorem* [1, 2] *we have a weak homotopy equivalence*

$$\mathbb{Z} \times BGL \simeq \Omega^2(\mathbb{Z} \times BGL);$$

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thus BGL is an infinite loop space. Similar results hold for BO and BSp, where O and Sp are the infinite orthogonal and symplectic group over $\mathbb{C}$ respectively.

In fact we have a very general method of construction. First, we need some preliminaries.

Let Σ_n be the symmetric group on the set $\{1, 2, \cdots, n\}$. For any $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$, $\sigma \oplus \tau$ is given by $\sigma \oplus \tau =$

$$\sigma \oplus \tau(i) = \begin{cases} \sigma(i), & 1 \leq i \leq m, \qquad \text{(1a)} \\ m + \tau(i), & m < i \leq m+n, \qquad \text{(1b)} \end{cases}$$

and $c(m,n) \in \Sigma_{m+n}$ is defined by

$$c(m,n)(i) = \begin{cases} n + i, & 1 \leq i \leq m, \qquad \text{(2a)} \\ i - m, & m < i \leq m+n. \qquad \text{(2b)} \end{cases}$$

The definitions imply $c(m,n) = c(n,m)^{-1}$.

Theorem 1.2. *Given a sequence of topological groups $G(1), G(2), \cdots, G(n), \cdots$ together with homomorphisms $\phi_m : \Sigma_m \to G(m)$, $f_m : G(m) \to G(m+1)$, $m > 0$, satisfying,*
1) for any $\alpha \in \Sigma_m$, we have $f_m \phi_m(\alpha) = \phi_{m+1}(\alpha)$;
2)set $f_{m,n} := f_{m+n-1} \cdots f_{m+1} f_m$, then $\phi_n(c(n,m))(f_{n,m}(G(n)))\phi_n(c(m,n))$ and $f_{m,n}(G(m))$ are commutative in $G(m+n)$;
3)let $G = \lim_{n\to\infty} G(n)$ and let $\pi' = [\pi, \pi]$ be the commutator subgroup of $\pi = \pi_0(G)$, we have $\pi' = [\pi', \pi']$.
Then BG^+ (where $+$ means the Quillen's plus construction for BG and $\pi' \subseteq \pi_1(BG)$) is an infinite loop space.

Proof. Define a topological category ξ as follows. The objects of ξ are non-negative integers, $hom_\xi(m,n)$ is empty if $m \neq n$ and $hom_\xi(m,m) = G(m)$. One checks that $(\xi, \oplus, 0, c)$ has a structure of permutative category, M is the corresponding classifying space. The rest of the proof is the same as in [3]p.62 □

Corollary 1.3. *Let R be a commutative ring and set*

$$SL(\infty, R) = \lim_{n\to\infty} SL(n, R),$$

then $BSL(\infty, R)^+$ is an infinite loop space.

Proof. We can easily find natural homomorphisms $\phi_n : \Sigma_n \to SL(2n, R)$, $n > 0$ that satisfy the conditions of the Theorem 1.2. □

Similarly we can show that $BGL(\infty, R)^+$, $BO(\infty, R)^+$, $BSO(\infty, R)^+$, $BSp(\infty, R)^+$ are all infinite loop spaces. In fact they are functors from the category of commutative rings to the category of infinite loop spaces. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups, which are infinite dimensional generalization of algebraic groups. Roughly speaking, there are seven infinite classes of affine Kac-Moody groups, and to each infinite class we can associate an analogous functor.

This paper is structured as follows. §2 is a short review of Kac-Moody algebras and Kac-Moody groups, in §3 we construct the infinite loop spaces corresponding to affine Kac-Moody groups of type $A^{(2)}_{2l-1}$, in the final section we consider several variations and the other cases. Throughout this paper R will be a fixed commutative ring.

2. Kac-Moody Algebras and Kac-Moody Groups

In this section, we give a brief review of Kac-Moody algebras and Kac-Moody groups, details can be found in [4, 6, 7].

Definition 2.1. *A generalized Cartan matrix is a matrix $A = (a_{i,j})^n_{i,j=1}$ satisfying, $a_{i,i} = 2$, $a_{i,j}$ are non-positive integers for $i \neq j$, and $a_{i,j} \neq 0$ implies $a_{j,i} \neq 0$.*

Definition 2.2. *The Kac-Moody algebra $\mathfrak{g}(A)$ associated to a generalized Cartan matrix $A = (a_{ij})^n_{i,j=1}$ is the Lie algebra (over $\mathbb{C}$) generated by $3n$ elements e_i, f_i, h_i, $(i = 1, \cdots, n)$ with the following defining relations:*

$$[h_i, h_j] = 0;\ [h_i, e_j] = a_{ij}e_j;\ [h_i, f_j] = -a_{ij}f_j; [e_i, f_j] = \delta_{i,j}h_i;$$

$$(ad\ e_i)^{1-a_{ij}}e_j = 0,\ (ad\ f_i)^{1-a_{ij}}f_j = 0,\ if\ i \neq j.$$

Let $A = (a_{i,j})^n_1$ be a generalized Cartan matrix. For $0 < i, j \leq n$ set $m_{i,j} = 2, 3, 4$ *or* 6 if $a_{i,j}a_{j,i} = 0, 1, 2$ *or* 3 respectively and set $m_{i,j} = 0$ otherwise. We associate to A a discrete group $W(A)$ (the $Weyl\ group$) on n generators $s_1, \cdots, s_n$ with relations $\{(s_is_j)^{m_{i,j}} = 1\}_{0<i,j\leq n}$.

As $ad\ e_i$ and $ad\ f_i$ are locally nilpotent endomorphisms of $\mathfrak{g}(A)$, the expressions $exp(e_i) = \sum_{n\geq 0} \frac{(ad\ e_i)^n}{n!}$ and $exp(f_i) = \sum_{n\geq 0} \frac{(ad\ f_i)^n}{n!}$ make sense.

Set $s_i' = exp(e_i)exp(-f_i)exp(e_i) \in Aut(\mathfrak{g}(A))$ and let $W'(A)$ be the subgroup of $Aut(\mathfrak{g}(A))$ generated by the s_i'. The map $s_i' \to s_i$ extends to a group homomorphism $\phi : W'(A) \to W(A)$.

Let V be the vector space over $\mathbb{Q}$, with basis $\{a_i\}_{i=1,\cdots,n}$ and let $W(A)$ act on V by $s_i(a_j) = a_j - a_{i,j}a_i$. *Real roots* of $A = (a_{i,j})_1^n$ are defined to be elements of V of the form $w(a_i)$, with $w \in W(A)$ and $0 < i \leq n$. Each real root a is an integral linear combination of $\{a_i\}$, the coefficients of which of all positive or negative; the real root a is said to be *positive* or *negative* accordingly. Denote by $\triangle$, $\triangle_+$, $\triangle_-$ the sets of all real roots, positive and negative real roots respectively. We say that a set of real roots θ is *prenilpotent* if there exist $w, w' \in W(A)$ such that all elements of $w(\theta)$ are positive and all elements of $w'(\theta)$ are negative; if, moreover, $a, b \in \theta$ and $a + b \in \triangle$ imply $a + b \in \theta$, then we said that θ is *nilpotent*.

For $0 < i \leq n$ and $w' \in W'(A)$, the pair of opposite elements $w'\{e_i, -e_i\} \subset \mathfrak{g}(A)$ depends only on the real root $a = \phi(w')(a_i)$ (see [7] for the proof of this claim); set $E_a = w'\{e_i, -e_i\}$ and denote by L_a the $\mathbb{C}$-subalgebra of $\mathfrak{g}(A)$ generated by E_a.

For each real root a, we denote by $\mathfrak{U}_a$ the group scheme over $\mathbb{Z}$ isomorphic to *Spec* $\mathbb{Z}$ and whose Lie algebra is the $\mathbb{Z}$-subalgebra of $\mathfrak{g}(A)$ generated by E_a.

Let θ be a nilpotent set of real roots, then $L_\theta = \bigoplus_{a\in\theta} L_a$ is a nilpotent Lie algebra. Let U_θ be the unipotent complex algebraic group whose Lie algebra is L_θ. The following proposition was proved in [7].

Proposition 2.3. *There exist a uniquely defined group scheme $\mathfrak{U}_\theta$ over $\mathbb{Z}$ containing all $\mathfrak{U}_a$ for $a \in \theta$, whose fibre over $\mathbb{C}$ is the group U_θ and such that for any order on θ, the product morphism $\prod_{a\in\theta} \to \mathfrak{U}_\theta$ is an isomorphism of the underlying schemes.*

Now we present Tits' definition of Kac-Moody group associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j=1}^n$ and a commutative ring R.

Let $\wedge$ be a free abelian group with basis $h_1, \cdots, h_n$, and $\wedge'$ its dual, then there are n elements $\alpha_1, \cdots, \alpha_n \in \wedge'$ satisfying $\langle h_i, \alpha_j\rangle = a_{i,j}$. Set $\mathfrak{T}(R) = Hom(\wedge', R^*)$. The group $W(A)$ also acts on $\wedge'$ by $s_i(\lambda) = \lambda - \langle\lambda, h_i\rangle\alpha_i$. The automorphism of $\mathfrak{T}(R)$ induced by s_i will also denoted by s_i.

For a real root a, and a nilpotent set of real roots θ, set $\mathfrak{U}_a(R)$, $\mathfrak{U}_\theta(R)$ to be the groups of R points of $\mathfrak{U}_a \times Spec\ R$ and $\mathfrak{U}_\theta \times Spec\ R$ respectively. For each pair of roots $\{a, b\}$, set $\vartheta(a, b) = (\mathbb{N}a + \mathbb{N}b) \cap \triangle$.

The *Steinberg group* $\mathfrak{S}(R)$ over R is defined as the inductive limit of the groups $\mathfrak{U}_a(R)$ and $\mathfrak{U}_{\vartheta(a,b)}(R)$, where $a \in \triangle$ and $\{a, b\}$ runs over all prenilpotent pairs of roots, relative to all the canonical injections $\mathfrak{U}_c(R) \to \mathfrak{U}_{\vartheta(a,b)}(R)$ for $c \in \vartheta(a, b)$. For each $0 < i \leq n$, $s'_i := exp(e_i)exp(-f_i)exp(e_i)$ is an automorphism of $\mathfrak{g}(A)$ which permutes the L_a and the E_a; therefore, it induces an automorphism of $\mathfrak{S}(R)$ which we again denote by s'_i.

Remark 2.4. *For any a, b in a nilpotent set θ and any $r, r' \in R$, the following commutation relation holds inside $\mathfrak{U}_\theta(R)$:*

$$[x_a(r), x_b(r')] = \prod_{c=ma+nb} x_c(k(a, b; c)r^m r'^n),$$

where $c = ma + nb$ runs over $\vartheta(a, b) - \{a, b\}$ and $x_a : R \to \mathfrak{U}_a(R)$, $x_b : R \to \mathfrak{U}_b(R)$ denote the isomorphisms associated to a and b.

Definition 2.5. *The Kac-Moody group $G_A(R)$ associated to A over R is defined to be the quotient of the free product of $\mathfrak{S}(R)$ and $\mathfrak{T}(R)$ by the following relations.*

$$tx_i(r)t^{-1} = x_i(t(\alpha_i)r); \quad \widetilde{s}_i t \widetilde{s}_i^{-1} = s'_i(t);$$
$$\widetilde{s}_i(r^{-1}) = \widetilde{s}_i r^{h_i} \text{ for } r \in R^* \quad \widetilde{s}_i u \widetilde{s}_i^{-1} = s'_i(u),$$

where t is an element from $\mathfrak{T}(R)$, r is an invertible element of R, $x_i : R \to \mathfrak{U}_{a_i}(R)$ and $x_{-i} : R \to \mathfrak{U}_{-a_i}(R)$ are the isomorphisms associated to e_i and f_i respectively, $\widetilde{s}_i(r)$ is the canonical image of $x_i(r)x_{-i}(r^{-1})x_i(r)$ in $\mathfrak{S}(R)$, $\widetilde{s}_i = \widetilde{s}_i(1)$, and $r^{h_i} \in \mathfrak{T}(R)$ is defined by $r^{h_i}(\lambda) = r^{\langle\lambda,h_i\rangle}$ for $\lambda \in \wedge'$.

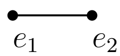
It is easy to see $G_A(R)$ is functorial in R, we call G_A the *Tits functor* associated to $A = (a_{ij})^n_{i,j=1}$. Set $r = 1$ in $\widetilde{s}_i(r^{-1}) = \widetilde{s}_i r^{h_i}$ we have $\widetilde{s}_i^2 = (-1)^{h_i}$, this formula will be used in the next section.

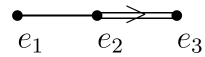
Remark 2.6. *The above defining relations was given in* [6]*, and is slightly different from that of* [7]*, in fact the formula $\widetilde{s}_i^2 = (-1)^{h_i}$ cannot be derived from the defining relations in* [7].

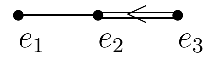
Remark 2.7. *From the defining relations we see that $G_A(R)$ (as a group) is generated by the image of $\mathfrak{U}_{a_i}(R)$ in $G_A(R)$.*

In §3 we need the following lemma.

Lemma 2.8. *Let A be a Cartan matrix of type*

A_2 

, B_3 

or C_3 

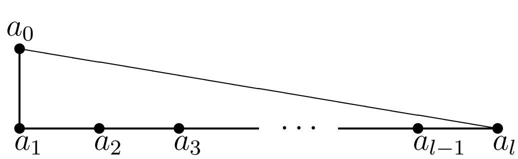
respectively, then the corresponding Kac-Moody group satisfies $G_A(R) = [G_A(R), G_A(R)]$.

Proof. In the case of A_2, we have the commutation relation $[x_{e_1}(1), x_{e_2}(r)] = x_{e_1+e_2}(r)$, hence the image of $\mathfrak{U}_{e_1+e_2}(R)$ lies in $[G_A(R), G_A(R)]$. But the Weyl group acts transitively on the real roots, hence the image of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ lies in $[G_A(R), G_A(R)]$ too. Thus by Remark 2.7, we have $G_A(R) = [G_A(R), G_A(R)]$.

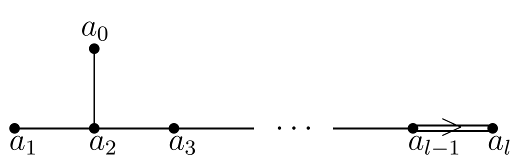
In the case of C_3, the above proof shows that the image of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ lies in $[G_A(R), G_A(R)]$. A direct computation shows that in $\mathfrak{U}_{\vartheta(e_2,e_3)}(R)$ we have $[x_{e_3}(r), x_{e_2}(1)] = x_{e_2+e_3}(-r)x_{e_2+2e_3}(-r)$, As the Weyl group acts transitively on the set of loot roots, we have $\mathfrak{U}_{e_2+2e_3}(R)$ lies in $[G_A(R), G_A(R)]$ and so is $\mathfrak{U}_{e_2+e_3}(R)$. But the Weyl group acts transitively on the set of short roots too, hence $\mathfrak{U}_{e_3}(R)$ also lies in $[G_A(R), G_A(R)]$. By Remark 2.7 again, we have $G_A(R) = [G_A(R), G_A(R)]$. The proof for the case of B_3 is similar. □

3. Construction of infinite loop spaces associated to $A^{(2)}_{2l-1}$

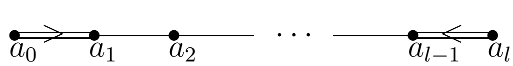
As shown in [4] there are seven infinite classes of generalized Cartan matrices of affine type, whose Dynkin diagrams are listed below.



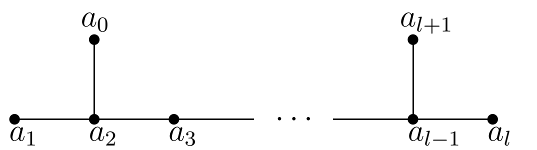
$A^{(1)}_l$



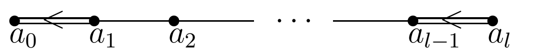
$B^{(1)}_l$



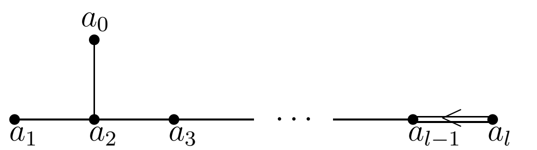
$C^{(1)}_l$



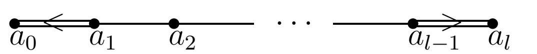
$D_{l+1}^{(1)}$



$A_{2l}^{(1)}$



$A_{2l-1}^{(2)}$



$D_{l+1}^{(2)}$

To each infinite class and any commutative ring R we want to associate a sequence of Kac-Moody groups $G(n)$ that satisfies the conditions of Theorem 1.2. First consider the case of $A_{2l-1}^{(2)}$, let $\mathfrak{g}_l$ and $G_l(R)$ be the corresponding Kac-Moody algebra and group respectively. In the following we use the notations of §2 freely, sometimes the subscript l will be used to indicate that the notations are associated to $A_{2l-1}^{(2)}$. For example, V_l will be the vector space over $\mathbb{Q}$, with basis $\{a_i\}_{i=0,\cdots,l}$. The group $W_l(A)$ acts on V_l and $\triangle_l$ denotes the set of real roots of $A_{2l-1}^{(2)}$

In $\mathfrak{g}_{l+1}$ set $e_l' = s_l'(e_{l+1})$, $f_l' = s_l'(f_{l+1})$, $h_l' = s_l'(h_{l+1}) = h_{l+1} + h_l$ respectively and for $i < l$ set $e_i' = e_i$, $f_i' = f_i$, $h_i' = h_i$ respectively.

Lemma 3.1. *In $\mathfrak{g}_{l+1}$ we have, for $i, j \leq l$,*

$$[h_i', h_j'] = 0;\ [h_i', e_j'] = a_{ij}e_j';\ [h_i', f_j'] = -a_{ij}f_j'; [e_i', f_j'] = \delta_{i,j}h_i';$$

$$(ad\ e_{l-1})^3 e_l' = 0;\ (ad\ f_{l-1})^3 f_l' = 0.$$

Proof. The first four relations follow from direct computations. Now set $\mathfrak{g}_{l-1} = \mathbb{C}e_{l-1} \oplus \mathbb{C}f_{l-1} \oplus \mathbb{C}h_{l-1}$ and consider $\mathfrak{g}_{l+1}$ as a $\mathfrak{g}_{l-1}$-module by restricting of the adjoint representation. Since $[h_{l-1}, e_l'] = -2e_l'$ and $[f_{l-1}, e_l'] = 0$ (follows from the fact that every root is either positive or negative), the

representation theory of $\mathfrak{g}_0 \cong sl_2(\mathbb{C})$ implies $(ad\ e_{l-1})^3 e'_l = 0$. The proof for the last relation is exactly the same. □

By the defining relations of $\mathfrak{g}_l$, the map $e_i \to e'_i$, $f_i \to f'_i$ extends to an injective Lie algebra homomorphism $\varphi_l : \mathfrak{g}_l \to \mathfrak{g}_{l+1}$.

Lemma 3.2. *Define a linear map $\tau_l : V_l \to V_{l+1}$ by $\tau_l(a_i) = a_i$ for $i < l$ and $\tau_l(a_l) = 2a_l + a_{l+1}$, then $\tau_l(\triangle_l^{\pm}) \subset \triangle_{l+1}^{\pm}$ and $\varphi_l(E_a) = E_{\tau_l(a)}$ for any $a \in \triangle_l$.*

Proof. It is easy to see that the map $s_i \to s_i$ for $i < l$ and $s_l \to s_l s_{l+1} s_l$ extends to a group homomorphism $w_l : W_l(A) \to W_{l+1}(A)$ and for any $v \in V_l$ and $W \in W_l(A)$ we have $\tau_l \cdot W(v) = w_l(W) \cdot \tau_l(v)$. Thus the first assertion follows readily. Similarly, the map $s'_i \to s'_i$ for $i < l$ and

$$s'_l \to s'_l s'_{l+1} (s'_l)^{-1} = exp(e'_i) \cdot exp(-f'_i) \cdot exp(e'_i)$$

extends to a group homomorphism $w'_l : W'_l(A) \to W'_{l+1}(A)$, where $W'_l(A) \subseteq Aut(\mathfrak{g}(A)_l)$ and $W'_{l+1}(A) \subseteq Aut(\mathfrak{g}(A)_{l+1})$. One checks that w_l and w'_l are compatible with the homomorphisms $\phi_l : W'_l(A) \to W_l(A)$ and $\phi_{l+1} : W'_{l+1}(A) \to W_{l+1}(A)$. We also have for any $\omega' \in W'_l(A)$, $\varphi_l \cdot \omega' = w'_l(\omega') \cdot \varphi_l$. Now we are ready to prove the second assertion. First, it is true for $a = a_i$, $i \leq l$ by the definition of φ_l. Let $a = \phi_l(\omega')(a_i)$ be an element of $\triangle_l$, with $\omega' \in W'_l(A)$, then $\varphi_l(E_a) = \varphi_l \omega'(E_{a_i}) = w'_l(\omega')\varphi_l(E_{a_i}) = w'_l(\omega')(E_{\tau_l(a_i)}) = E_{\phi_{l+1} w'_l(\omega')(\tau_l(a_i))} = E_{w_l \phi_l(\omega')(\tau_l(a_i))} = E_{\tau_l(\phi_l(\omega')(a_i))} = E_{\tau_l(a)}$. This finishes the proof. □

For any $a \in \triangle_l$, let $\mathfrak{U}_a$ be the corresponding group scheme defined in §2, then we can define a homomorphism $\psi_a : \mathfrak{U}_a \to \mathfrak{U}_{\tau_l(a)}$ that is compatible with the map $E_a \to E_{\tau_l(a)}$.

Lemma 3.3. *Let $\theta \subset \triangle_l$ be a nilpotent set of real roots, then $\tau_l(\theta) \subset \triangle_{l+1}$ is also nilpotent; let $\mathfrak{U}_\theta$ and $\mathfrak{U}_{\tau_l(\theta)}$ be the group schemes in Proposition 2.3, then the homomorphism $\psi_a : \mathfrak{U}_a \to \mathfrak{U}_{\tau_l(a)}$ for $a \in \theta$ extends uniquely to a homomorphism $\psi_\theta : \mathfrak{U}_\theta \to \mathfrak{U}_{\tau_l(\theta)}$.*

Proof. By lemma 3.2 the homomorphism $\varphi_l : \mathfrak{g}_l \to \mathfrak{g}_{l+1}$ induces an isomorphism $L_\theta \to L_{\tau_l(\theta)}$. Thus for $a, b \in \theta$, the commutation relation of $\mathfrak{U}_a$ and $\mathfrak{U}_b$ in $\mathfrak{U}_\theta$ is exactly the same as that of $\mathfrak{U}_{\tau_l(a)}$ and $\mathfrak{U}_{\tau_l(b)}$ in $\mathfrak{U}_{\tau_l(\theta)}$. Now the lemma follows readily. □

By Lemma 3.2 and Lemma 3.3 the group homomorphisms $\psi_a(R) : \mathfrak{U}_a(R) \to \mathfrak{U}_{\tau_l(a)}(R)$, $a \in \triangle_l$, extend to a group homomorphism $\psi(R) : \mathfrak{G}_l(R) \to \mathfrak{G}_{l+1}(R)$.

Let $\wedge_l$ be a free abelian groups with basis $h_0, \cdots, h_l$ and $\wedge'_l$ its dual. Define linear map $\omega_l : \wedge_l \to \wedge_{l+1}$ by $\omega_l(h_i) = h_i$ for $i < l$ and $\omega_l(h_l) = h_l + 2h_{l+1}$. Denote by ω'_l the dual map of ω_l, then ω'_l induces a group homomorphism $\omega_l(R) : \mathfrak{T}_l(R) \to \mathfrak{T}_{l+1}(R)$.

From the defining relations of Kac-Moody groups and the constructions of $\psi(R)$ and $\omega_l(R)$ we see that the homomorphism of free products $\psi * \omega_l(R) : \mathfrak{S}_l(R) * \mathfrak{T}_l(R) \to \mathfrak{S}_{l+1}(R) * \mathfrak{T}_{l+1}(R)$ reduces to a homomorphism $g_l : G_l(R) \to G_{l+1}(R)$. Set $G(n) := G_{2n}(R)$ and $f_n := g_{2n+1} \cdot g_{2n}$. In order to apply Theorem 1.2, we have to define group homomorphism $\varsigma_n : \Sigma_n \to G(n)$ for each $n > 0$.

First we need some preliminaries. Let $\overline{W}_l$ be the signed permutation group, i.e., the group of linear transformations of $\mathbb{R}^l$ leaving invariant the set $\{\pm e_i\}$ of standard basis vectors and their negatives. It has $l-1$ generators $\overline{r}_1, \cdots, \overline{r}_{l-1}$ and the following defining relations:

$$\overline{r}_j \overline{r}_i^2 \overline{r}_j^{-1} = \overline{r}_i^2 \overline{r}_j^{-2a_{i,j}}$$

$$\overline{r}_i \overline{r}_j \overline{r}_i \cdots = \overline{r}_j \overline{r}_i \overline{r}_j \cdots (m_{i,j} \ \textit{factors on each side}),$$

where $\overline{r}_i$ is defined by sending $\{e_i, e_{i+1}\}$ to $\{-e_{i+1}, e_i\}$ and leaves the other basis vectors invariant.

Lemma 3.4. *The $\widetilde{s}_i$, $0 < i < l$ in $G_l(R)$ satisfy the following two relations,*

$$\widetilde{s}_j \widetilde{s}_i^2 \widetilde{s}_j^{-1} = \widetilde{s}_i^2 \widetilde{s}_j^{-2a_{i,j}},$$

$$\widetilde{s}_i \widetilde{s}_j \widetilde{s}_i \cdots = \widetilde{s}_j \widetilde{s}_i \widetilde{s}_j \cdots (m_{i,j} \ \textit{factors on each side}).$$

Let $\widetilde{W}_l$ be the subgroup of $G_l(R)$ generated by $\{\widetilde{s}_i\}_{0<i<l}$, then the map $\overline{r}_i \to \widetilde{s}_i$ extends to a group homomorphism $h_l : \overline{W}_l \to \widetilde{W}_l$.

Proof. We prove the first assertion and the second assertion will follow directly. As $\widetilde{s}_i^2 = (-1)^{h_i}$ the first relation is equivalent to

$$\widetilde{s}_j (-1)^{h_i} \widetilde{s}_j^{-1} = (-1)^{h_i - 2a_{i,j} h_i},$$

which is one of the defining relations of $G_l(R)$. The second relation was proved in Remark 3.7 of [7]. □

Define $w_i \in \overline{W}_{2n}$ by sending $\{e_{2i-1}, e_{2i}\}$ to $\{e_{2i+1}, e_{2i+2}\}$ and leaving the other basis vectors invariant. set $S_i = h_{2n}(w_i)$, direct computation shows that $S_i = \widetilde{s}_{2i+1}^3 \widetilde{s}_{2i} \widetilde{s}_{2i-1} \widetilde{s}_{2i+1} \widetilde{s}_{2i} \widetilde{s}_{2i-1}$.

Let $\sigma(i) \in \Sigma_n$ be the permutation that swaps the i-th element with the $(i+1)$-th one, then the map $\sigma(i) \to S_i$ extends to a group homomorphism $\varsigma_n : \Sigma_n \to G(n) = G_{2n}(R)$.

Theorem 3.5. *let $G = \lim_{n\to\infty} G_n(R)$, then $\pi = \pi_0(G)$ satisfies $\pi = [\pi, \pi]$. Applying Quillen's plus construction toBG and $\pi^{'} \subseteq \pi_1(BG)$, we get an infinite loop space BG^+.*

Proof. The first assertion follows directly from Lemma 2.8. In order to apply Theorem 1.2 to $G(n) = G_{2n}(R)$, $f_n = g_{2n+1}g_{2n} : G(n) \to G(n+1)$ and $\varsigma_n : \Sigma_n \to G(n) = G_{2n}(R)$, we only need to verify the condition 2) of Theorem 1.2. Set $f_{m,n} = f_{m+n-1}\cdots f_{m+1}f_m$, we want to show that $f_{m,n}(G(m))$ and $c(n,m)(f_{n,m}(G(n)))c(m,n)$ are commutative in $G(m+n)$. Set $s_{nm} := \phi_{2m+2n}\varsigma_{m+n}(c(n,m))$ in the following, recall that ϕ_{2m+2n} is the natural homomorphism $W'(A^{(2)}_{4m+4n-1}) \to W(A^{(2)}_{4m+4n-1})$.
By remark 2.7, $f_{m,n}(G(m))$ is generated by the subgroups $\{\mathfrak{U}_a(R)\}_{a\in\Theta}$ and $c(n,m)(f_{n,m}(G(n)))c(m,n)$ is generated by the subgroups $\{\mathfrak{U}_a(R)\}_{a\in\Theta'}$, where

$$\begin{aligned}\Theta &= \{\pm a_0, \cdots, \pm a_{2m-1}, (s_{2m-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n})\}\\ &= \{\pm a_0, \cdots, \pm a_{2m-1}, \pm(2a_{2m-1} + \cdots + 2a_{2m+2n-1} + a_{2m+2n})\}\end{aligned}$$

and

$$\Theta' = s_{n,m}\{\pm a_0, \cdots, \pm a_{2n-1}, (s_{2n-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n})\}.$$

Thus in order to verify condition (2) it suffices to show that for any $\alpha \in \Theta$ and $\beta \in \Theta'$, $\mathfrak{U}_\alpha(R)$ and $\mathfrak{U}_\beta(R)$ are commutative, but this can be deduced from the fact that the subalgebras $L_{\pm\alpha}$ and $L_{\pm\beta}$ of $\mathfrak{g}_{2m+2n}$ are commutative. Indeed, when $L_{\pm\alpha}$ and $L_{\pm\beta}$ are commutative, one checks that $\{\alpha, \beta\}$ is a prenilpotent pair and $\vartheta(a,b) = \{\alpha, \beta\}$, hence by Remark 2.4 the group $\mathfrak{U}_{\vartheta(a,b)}(R)$ is commutative. Thus in order to finish the proof it suffices to show that for any $\alpha \in \Theta$ and $\beta \in \Theta'$, $L_{\pm\alpha}$ and $L_{\pm\beta}$ are commutative.

Direct computation shows that

$$\begin{aligned}&(s_{2m-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{nm}(\pm a_{2m+2n});\\ &(s_{2n-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{mn}(\pm a_{2m+2n});\\ &s_{mn}(\pm a_0) = \pm(a_0 + a_1 + 2(a_2 + \cdots + a_{2m}) + a_{2m+1});\\ &s_{nm}\{\pm a_1, \cdots, \pm a_{2n-1}\} = \{\pm a_{2m+1}, \cdots, \pm a_{2m+2n-1}\};\\ &s_{nm}\{\pm a_{2n+1}, \cdots, \pm a_{2m+2n-1}\} = \{\pm a_1, \cdots, \pm a_{2m-1}\}.\end{aligned}$$

Thus we only need to show that $L_{\pm(a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1})}$ is commutative with $L_{\pm a_0}$ and $L_{\pm a_{2m+2n}}$ is commutative with $L_{\pm(2a_{2m-1}+\cdots+2a_{2m+2n-1}+a_{2m+2n})}$. We proof the first assertion, the proof for the second one is similar.

Firstly, we have $[f_0, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] \in L_{a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}$, but it is well known that the highest root in $\mathbb{Z}a_1+\mathbb{Z}a_2+\cdots+\mathbb{Z}a_{2m+1}\cap\triangle_{2m+2n}$ is $a_1+\cdots+a_{2m+1}$. Hence $[f_0, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}]=0$. We also have $[h_0, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}]=0$. Set $\mathfrak{g}_0=\mathbb{C}e_0\oplus\mathbb{C}f_0\oplus\mathbb{C}h_0$ and consider $\mathfrak{g}_{2m+2n}$ as a $\mathfrak{g}_0$-module by restricting of the adjoint representation. By the representation theory of $\mathfrak{g}_0\cong sl_2(\mathbb{C})$, it follows that

$$[e_0, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}]=0.$$

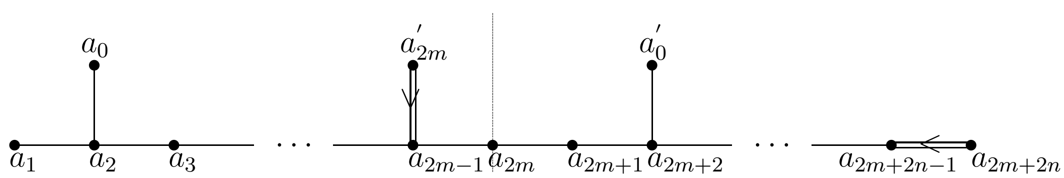
Similarly we have

$$[e_0, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}]=0$$

and

$$[f_0, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}]=0.$$

This finishes the proof of the theorem. The following Dynkin diagram would illustrate our proof, where $a_0^{'}$ and $a_{2m}^{'}$ are $2a_{2m-1}+\cdots+2a_{2m+2n-1}+a_{2m+2n}$ and $s_{n,m}(a_0)$ respectively.



□

4. The constructions in the other cases

The constructions in the other cases is similar. For example in the case of $A_l^{(1)}$, let $\mathfrak{g}_l$ be the Kac-Moody algebra associated to $A_l^{(1)}$, and in $\mathfrak{g}_{l+1}$ set $e_l'=s_l'(e_{l+1})$, $f_l'=s_l'(f_{l+1})$, $h_l'=s_l'(h_{l+1})=h_{l+1}+h_l$ respectively and for $i<l$ set $e_i'=e_i$, $f_i'=f_i$, $h_i'=h_i$ respectively. In the case of $D_{l+1}^{(1)}$, set $e_l'=s_l'\cdot s_{l-1}'(e_{l+1})$, $f_l'=s_l'\cdot s_{l-1}'(f_{l+1})$, $h_l'=s_l'\cdot s_{l-1}'(h_{l+1})=h_{l+1}+h_l+h_{l-1}$ respectively. For the rest constructions we just repeat the arguments of the previous section.

Remark 4.1. *In §3 we require that $\wedge_l$ is freely generated by $\{h_0,\cdots,h_l\}$, in fact this restriction is not necessary. For example, in the case of $A_l^{(1)}$ we can set $\wedge_l$ to be freely generated by $\{h_1,\cdots,h_l\}$ and add an $h_0:=-h_1-\cdots-h_l$. When R is a field K, the corresponding Kac-Moody group $G_l(K)$ is isomorphic to $SL_{l+1}(K[t,t^{-1}])$, then $G(\infty,K)^+$ is of course an infinite*

loop space. However, we don't know the explicit realization of $G_l(R)$ in the general case.

We can also treat the (topological) affine Kac-Moody groups over $\mathbb{C}$ (see [5] for the definition), and applying the method of §2 we have the following result.

Theorem 4.2. *Let $\{A_l\}_{l>2}$ be one of the seven (infinite) classes of affine generalized Cartan matrices and let $\{G_l\}_{l>2}$ be the associated simply-connected Kac-Moody groups over $\mathbb{C}$, then we can define for each $l > 2$ a natural homomorphism $f_l : G_l \to G_{l+1}$ such that $BG = \lim_{l\to\infty} BG_l$ is an infinite loop space.*

Remark 4.3. *In fact there exists a (infinite) classes of classical Lie groups $\{G(l)\}_{l>2}$ such that G_l is isomorphic to a central extension of the group of polynomial loops or twisted polynomial loops on $G(l)$.*

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